

## ON SUMS OF RUDIN-SHAPIRO COEFFICIENTS II

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Let  $\{a(n)\}$  be the Rudin-Shapiro sequence, and let  $s(n) = \sum_{k=0}^n a(k)$  and  $t(n) = \sum_{k=0}^n (-1)^k a(k)$ . In this paper we show that the sequences  $\{s(n)/\sqrt{n}\}$  and  $\{t(n)/\sqrt{n}\}$  do not have cumulative distribution functions, but do have logarithmic distribution functions (given by a specific Lebesgue integral) at each point of the respective intervals  $[\sqrt{3/5}, \sqrt{6}]$  and  $[0, \sqrt{3}]$ . The functions  $a(x)$  and  $s(x)$  are also defined for real  $x \geq 0$ , and the function  $[s(x) - a(x)]/\sqrt{x}$  is shown to have a Fourier expansion whose coefficients are related to the poles of the Dirichlet series  $\sum_{n=1}^{\infty} a(n)/n^{\tau}$ , where  $\operatorname{Re} \tau > \frac{1}{2}$ .

**1. Introduction.** In this paper we are concerned with the Rudin-Shapiro sums

$$(1.1) \quad s(x) = \sum_{k=0}^{[x]} a(k),$$

$$(1.2) \quad t(x) = \sum_{k=0}^{[x]} (-1)^k a(k),$$

where the numbers  $a(k)$  are defined recursively by

$$(1.3) \quad a(2k) = a(k), \quad a(2k+1) = (-1)^k a(k), \quad k \geq 0, a(0) = 1.$$

An explicit formula for  $a(k)$  is given by

$$(1.4) \quad a(k) = (-1)^{e(k)},$$

where  $e(k) = \sum_{i=0}^{s-1} \epsilon_i \epsilon_{i+1}$  and  $k = \sum_{i=0}^s \epsilon_i 2^i$ ,  $\epsilon_i = 0$  or  $1$ . (See [1], Satz 1.)

The properties of these sums have been developed in [1], where it is shown that

$$(1.5) \quad \sqrt{\frac{3}{5}} < \frac{s(n)}{\sqrt{n}} < \sqrt{6},$$

$$(1.6) \quad 0 \leq \frac{t(n)}{\sqrt{n}} < \sqrt{3},$$

for  $n \geq 1$ , and that the sequences  $\{s(n)/\sqrt{n}\}$  and  $\{t(n)/\sqrt{n}\}$  are dense in the intervals  $[\sqrt{3/5}, \sqrt{6}]$  and  $[0, \sqrt{3}]$ .

Here we study the quotients  $s(n)/\sqrt{n}$  and  $t(n)/\sqrt{n}$  further by introducing the limit functions

$$\lambda(x) = \lim_{k \rightarrow \infty} \frac{s(4^k x)}{\sqrt{4^k x}},$$

$$\mu(x) = \lim_{k \rightarrow \infty} \frac{t(4^k x)}{\sqrt{4^k x}},$$

which are defined for  $x > 0$ . We show that  $\lambda(x)$  and  $\mu(x)$  are continuous functions of  $x$ , but are non-differentiable almost everywhere. Since  $\lambda$  and  $\mu$  satisfy the functional equations

$$(1.7) \quad \lambda(4x) = \lambda(x), \quad \mu(4x) = \mu(x),$$

the curves  $\{(x, \lambda(x)); 1 \leq x \leq 4\}$  and  $\{(x, \mu(x)); 1 \leq x \leq 4\}$  represent the limiting behavior of the quotients  $s(n)/\sqrt{n}$  and  $t(n)/\sqrt{n}$  on the intervals  $[4^k, 4^{k+1} - 1]$ , as  $k \rightarrow \infty$ . (See Figure 1 in §4.)

Equation (1.7) implies also that  $\lambda(x)$  has a Fourier series expansion of the form

$$(1.8) \quad \lambda(x) = \sum_{n=-\infty}^{\infty} c_n e^{\pi i n \log x / \log 2},$$

where  $c_n \in \mathbb{C}$ . This series is  $(C, 1)$  summable to  $\lambda(x)$  for all  $x > 0$ , and is convergent in the usual sense for almost all  $x > 0$ . In fact, we are able to give an explicit set on which (1.8) is convergent, the set of  $x > 0$  which are simply normal to the base 4. (See §4, 5, and [6].) This allows us to say, for example, that (1.8) converges when  $x = m + \frac{9}{85}$ , where  $m$  is a non-negative integer.

Formula (1.8) then leads to an explicit formula for  $s(x)$  of the form

$$(1.9) \quad s(x) = \sqrt{x} \sum_{n=-\infty}^{\infty} c_n x^{\pi i n / \log 2} + a(x), \quad x > 0,$$

where  $a(x)$  is an extension of the function  $a(n)$ , defined for real arguments  $x \geq 0$ . The function  $a(x)$  is bounded, and has an explicit representation in terms of the digits of  $x$  to the base 4. Formula (1.9) accounts for the roughly “periodic” behavior of the sequence  $\{s(n)/\sqrt{n}\}$ .

We show further that the Fourier coefficients  $c_n$  are related to the poles of the function  $\eta(\tau)$  defined by the Dirichlet series

$$\eta(\tau) = \sum_{n=1}^{\infty} \frac{a(n)}{n^{\tau}}, \quad \operatorname{Re} \tau > \frac{1}{2}.$$

This function has a meromorphic continuation to the whole complex plane, and its only poles in the half-plane  $\operatorname{Re} \tau > 0$  occur among the points  $\gamma_n = 1/2 + \pi ni / \log 2$ ,  $n \in \mathbf{Z}$ . We prove that  $\gamma_n c_n$  is equal to the residue of  $\eta(\tau)$  at  $\tau = \gamma_n$ , and use this fact to show that infinitely many of the points  $\gamma_n$  are poles of  $\eta(\tau)$ . This is seen to be a consequence of the fact that  $\lambda(x)$  is not everywhere differentiable.

Finally, we use  $\lambda(x)$  to prove the non-existence of the cumulative (or natural) distribution function of the sequence  $\{s(n)/\sqrt{n}\}$  on the interval  $(\sqrt{3/5}, \sqrt{6})$ . By this we mean the limit  $\lim_{x \rightarrow \infty} x^{-1} D(x, \alpha)$ , where  $\alpha \in (\sqrt{3/5}, \sqrt{6})$ , and  $D(x, \alpha)$  is the number of times  $s(n) \leq \alpha\sqrt{n}$  for  $1 \leq n \leq x$ .

In the positive direction, we prove that the logarithmic distribution function for  $\{s(n)/\sqrt{n}\}$ , defined to be

$$\lim_{x \rightarrow \infty} \frac{1}{\log x} \sum_{\substack{1 \leq r \leq x \\ s(r) \leq \alpha\sqrt{r}}} \frac{1}{r} = L(\alpha),$$

does exist for all  $\alpha \in [\sqrt{3/5}, \sqrt{6}]$ . We show that

$$L(\alpha) = \frac{1}{\log 4} \int_{E_\alpha} \frac{1}{x} dx,$$

where the integral is a Lebesgue integral and  $E_\alpha$  is the set  $E_\alpha = \{x: 1 \leq x \leq 4 \text{ and } \lambda(x) \leq \alpha\}$ . In other words,  $L(\alpha)$  is simply the (multiplicative) Haar measure of the set  $E_\alpha$ . There are similar results for  $\{t(n)/\sqrt{n}\}$ .

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**2. The functions  $\lambda(x)$  and  $\mu(x)$ .** We first prove the existence of the limit

$$(2.1) \quad \lambda(x) = \lim_{k \rightarrow \infty} \frac{s(4^k x)}{\sqrt{4^k x}}, \quad x > 0,$$

where  $s(x)$  is defined in (1.1). We will require the following formulas from [1] (see Satz 3), all of which hold for integers  $n \geq 0$ :

$$(2.2) \quad \begin{cases} s(4n) = 2s(n) - a(n), & s(4n+2) = 2s(n) + (-1)^n a(n), \\ s(4n+1) = 2s(n), & s(4n+3) = 2s(n). \end{cases}$$

We set  $\rho(d) = \chi(1 - d)$ , where  $\chi$  is the nontrivial character (mod 4), so that

$$(2.3) \quad \rho(d) = \begin{cases} 1, & \text{if } d \equiv 0 \pmod{4}, \\ -1, & \text{if } d \equiv 2 \pmod{4}, \\ 0, & \text{if } d \text{ is odd.} \end{cases}$$

Then, using (1.3), the relations in (2.2) can be written as the single formula

$$(2.4) \quad s(4n + d) = 2s(n) - \rho(d)a(4n + d), \quad n \geq 0, 0 \leq d \leq 3.$$

We will also need the 4-adic expansion of a non-negative real number  $x$ , namely

$$(2.5) \quad x = \sum_{r=0}^{\infty} d_r 4^{-r},$$

where the  $d_r$  are integers,  $0 \leq d_r \leq 3$  for  $r \geq 1$ , and infinitely many  $d_r$  are not equal to 3. We set

$$(2.6) \quad b_k = [4^k x] = \sum_{r=0}^k d_r 4^{k-r}$$

and note that

$$(2.7) \quad b_k = 4b_{k-1} + d_k, \quad \text{for } k \geq 1.$$

**THEOREM 1.** *The limit in (2.1) exists for all  $x > 0$ , and is given by the formula*

$$(2.8) \quad \lambda(x) = \frac{s(x)}{\sqrt{x}} - \frac{1}{\sqrt{x}} \sum_{r=1}^{\infty} \rho(d_r) a(b_r) 2^{-r}.$$

*Proof.* We have from (2.6), (2.7), and (2.4) that

$$\begin{aligned} s(4^k x) &= s([4^k x]) = s(b_k) = s(4b_{k-1} + d_k) \\ &= 2s(b_{k-1}) - \rho(d_k) a(b_k) \\ &= 2s(4^{k-1} x) - \rho(d_k) a(b_k), \end{aligned}$$

for  $k \geq 1$ . Continuing this reduction gives

$$(2.9) \quad s(4^k x) = 2^k s(x) - \sum_{r=1}^k \rho(d_r) a(b_r) 2^{k-r}, \quad \text{for } k \geq 1.$$

Hence

$$\frac{s(4^k x)}{\sqrt{4^k x}} = \frac{s(x)}{\sqrt{x}} - \frac{1}{\sqrt{x}} \sum_{r=1}^k \rho(d_r) a(b_r) 2^{-r}.$$

Equation (2.8) now follows by letting  $k \rightarrow \infty$ , since the series on the right side of (2.8) converges absolutely.

**COROLLARY.** *If  $n$  is a positive integer, then*

$$(2.10) \quad \lambda(n) = \frac{s(n-1)}{\sqrt{n}}.$$

*Proof.* In the notation of (2.5) and (2.6) we have that  $x = d_0 = n$ ,  $d_r = 0$  for  $r \geq 1$ , and  $b_k = 4^k n$ . Thus the infinite sum in (2.8) becomes

$$(2.11) \quad \sum_{r=1}^{\infty} \rho(d_r) a(b_r) 2^{-r} = \sum_{r=1}^{\infty} a(n) 2^{-r} = a(n),$$

and so

$$\lambda(n) = \frac{s(n)}{\sqrt{n}} - \frac{a(n)}{\sqrt{n}} = \frac{s(n-1)}{\sqrt{n}}. \quad \square$$

Equation (2.11) suggests the following extension of the function  $a(n)$ .

**DEFINITION.** For  $x \geq 0$ , set  $x = \sum_{r=0}^{\infty} d_r 4^{-r}$  as in (2.5), and define

$$(2.12) \quad a(x) = \sum_{r=1}^{\infty} \rho(d_r) a(b_r) 2^{-r},$$

where  $b_r = [4^r x]$  and  $\rho(d)$  is given by (2.3).

Using (2.12), we may now write (2.8) in the form

$$(2.13) \quad \lambda(x) = \{s(x) - a(x)\} x^{-1/2}, \quad x > 0.$$

We also note the functional equation

$$(2.14) \quad \lambda(4x) = \lambda(x), \quad x > 0,$$

which is an immediate consequence of (2.1).

**LEMMA 1.** *For  $k \geq 0$  and  $x > 0$  we have the estimate*

$$(2.15) \quad \left| \lambda(x) - \frac{s(4^k x)}{\sqrt{4^k x}} \right| \leq 2^{-k} x^{-1/2}.$$

*Proof.* It is clear from (2.12) that  $|a(x)| \leq 1$ . Thus, (2.13) implies

$$\left| \lambda(x) - \frac{s(x)}{\sqrt{x}} \right| \leq x^{-1/2}.$$

The lemma now follows on replacing  $x$  by  $4^k x$  and using (2.14).

LEMMA 2. (a) *If  $x > 0$ , then  $\lambda(x) \in [\sqrt{3/5}, \sqrt{6}]$ .*

(b) *The set  $\{\lambda(x) : x > 0\}$  is dense in  $[\sqrt{3/5}, \sqrt{6}]$ .*

*Proof.* For each  $x_1 \geq 1$ , equation (1.5) implies the inequalities

$$\begin{aligned} \sqrt{\frac{3}{5}} - x_1^{-1/2} &< \frac{s([x_1] + 1)}{\sqrt{[x_1] + 1}} - \frac{1}{\sqrt{x_1}} \\ &< \frac{s([x_1] + 1) - a([x_1] + 1)}{\sqrt{x_1}} = \frac{s(x_1)}{\sqrt{x_1}}, \end{aligned}$$

and

$$\frac{s(x_1)}{\sqrt{x_1}} \leq \frac{s([x_1])}{\sqrt{[x_1]}} < \sqrt{6}.$$

Now take  $x_1 = 4^k x$ , where  $k$  is chosen large enough so that  $x_1 \geq 1$ . Then the above estimates give

$$\sqrt{\frac{3}{5}} - 2^{-k} x^{-1/2} < \frac{s(4^k x)}{\sqrt{4^k x}} < \sqrt{6},$$

and letting  $k \rightarrow \infty$  proves (a).

We also note from (2.10) that  $\lambda(n) = s(n)/\sqrt{n} + o(1)$ . Thus (b) follows from the fact that the set  $\{s(n)/\sqrt{n} : n \geq 1\}$  is dense in  $[\sqrt{3/5}, \sqrt{6}]$ .

EXAMPLE. Let  $x = (3n + 2)/3$ , where  $n$  is an integer  $\geq 0$ . Then we have the expansion

$$x = n + \frac{2}{3} = n + \sum_{r=1}^{\infty} \frac{2}{4^r},$$

so  $d_0 = n$ ,  $d_r = 2$  and  $b_k = 4^k n + \sum_{r=0}^{k-1} 2 \cdot 4^r$  in the notation of (2.5) and (2.6). Using (1.4) it is easy to see that  $a(b_k) = (-1)^n a(n)$  for all  $k \geq 1$ . Thus (2.12) and (2.3) imply that

$$a(x) = \sum_{r=1}^{\infty} \rho(2) a(b_r) 2^{-r} = (-1)^{n+1} a(n),$$

so from (2.13),

$$\lambda\left(\frac{3n+2}{3}\right) = [s(n) + (-1)^n a(n)] \left(\frac{3}{3n+2}\right)^{1/2}.$$

In particular,

$$(2.16) \quad \lambda\left(\frac{2}{3}\right) = 2\sqrt{\frac{3}{2}} = \sqrt{6} \quad \text{and} \quad \lambda\left(\frac{5}{3}\right) = (2-1)\sqrt{\frac{3}{5}} = \sqrt{\frac{3}{5}}.$$

We now investigate the limit

$$(2.17) \quad \mu(x) = \lim_{k \rightarrow \infty} \frac{t(4^k x)}{\sqrt{4^k x}}, \quad x > 0,$$

where  $t(x)$  is defined in (1.2). For this we recall the elementary formula

$$(2.18) \quad t(n) = s(2n+1) - s(n), \quad n \geq 0,$$

from [1] (Satz 2).

**THEOREM 2.** *The limit in (2.17) exists for all  $x > 0$ . We have*

$$(2.19) \quad \mu(x) = \sqrt{2}\lambda(2x) - \lambda(x),$$

and

$$(2.20) \quad \mu(4x) = \mu(x).$$

*Proof.* From (2.18) it follows easily that

$$(2.21) \quad |t(x) - s(2x) + s(x)| \leq 1 \quad \text{for } x \geq 0.$$

Hence for any  $x > 0$ ,

$$\begin{aligned} \mu(x) &= \lim_{k \rightarrow \infty} \frac{t(4^k x)}{\sqrt{4^k x}} = \lim_{k \rightarrow \infty} \frac{s(2 \cdot 4^k x) - s(4^k x) + O(1)}{\sqrt{4^k x}} \\ &= \sqrt{2}\lambda(2x) - \lambda(x). \end{aligned}$$

Equation (2.20) follows immediately from (2.17).

**COROLLARY 1.** *For  $x > 0$ ,*

$$\lambda(x) = \sqrt{2}\mu(2x) + \mu(x).$$

*Proof.* Equations (2.19) and (2.14) imply that

$$\mu(2x) = \sqrt{2}\lambda(x) - \lambda(2x).$$

Multiplying through by  $\sqrt{2}$  and adding to (2.19) yields the result.

**COROLLARY 2.** *If  $n$  is a positive integer,  $\mu(n) = t(n-1)/\sqrt{n}$ .*

*Proof.* Immediate from (2.19), (2.10), and (2.18).

By virtue of (2.19), the function  $\mu(x)$  inherits its properties from  $\lambda(x)$ . In particular, we have

LEMMA 3. For  $k \geq 0$  and  $x > 0$ ,

$$(2.22) \quad \left| \mu(x) - \frac{t(4^k x)}{\sqrt{4^k x}} \right| \leq 3 \cdot 2^{-k} x^{-1/2}.$$

*Proof.* We see from (2.19), (2.21) and Lemma 1 (with  $k = 0$ ) that

$$\begin{aligned} & \left| \frac{t(x)}{\sqrt{x}} - \mu(x) \right| \\ &= \left| \frac{t(x) - s(2x) + s(x)}{\sqrt{x}} + \frac{s(2x)}{\sqrt{x}} - \sqrt{2} \lambda(2x) + \lambda(x) - \frac{s(x)}{\sqrt{x}} \right| \\ &\leq 3x^{-1/2}. \end{aligned}$$

The assertion (2.22) is therefore a consequence of this estimate and (2.20).

Just as in Lemma 2, one may use (2.17), (1.6), and Corollary 2 of Theorem 2 to prove

LEMMA 4. (a) If  $x > 0$ , then  $\mu(x) \in [0, \sqrt{3}]$ .

(b) The set  $\{\mu(x) : x > 0\}$  is dense in  $[0, \sqrt{3}]$ .

EXAMPLE. If  $x = (3n + 1)/3$ , then the expansion

$$x = n + \frac{1}{3} = n + \sum_{r=1}^{\infty} \frac{1}{4^r}$$

implies by (2.3) and (2.12) that  $a(x) = 0$ . Hence

$$\lambda\left(\frac{3n+1}{3}\right) = s(n) \left(\frac{3}{3n+1}\right)^{1/2}.$$

It follows from this and equations (2.19), (2.16) that

$$\mu\left(\frac{1}{3}\right) = \sqrt{3}, \quad \mu\left(\frac{2}{3}\right) = 0.$$

The examples of this section suggest that  $\sqrt{x}\lambda(x)$  is a rational number whenever  $x$  is. This is indeed true, as we shall now show.

THEOREM 3. If  $x > 0$  and  $x \in \mathbf{Q}$ , then  $\sqrt{x}\lambda(x) \in \mathbf{Q}$ .



*Proof.* By (2.13) it suffices to show that  $a(x) \in \mathbf{Q}$  if  $x \in \mathbf{Q}$ , since  $s(x) \in \mathbf{Z}$ . If  $x$  is rational, the 4-adic expansion of  $x$  must be ultimately periodic:

$$x = \sum_{r=0}^{\infty} d_r 4^{-r} \quad \text{where } d_{k+p} = d_k, k \geq k_0,$$

for some period length  $p$  and some  $k_0 \geq 1$ . To prove that  $a(x) \in \mathbf{Q}$  it is enough to prove that

$$(2.23) \quad \rho(d_{k+2p})a(b_{k+2p}) = \rho(d_k)a(b_k), \quad k \geq k_0,$$

by formula (2.12). Clearly  $\rho(d_{k+2p}) = \rho(d_k)$ ,  $k \geq k_0$ , and so we consider the term  $a(b_{k+2p})$ .

From (2.6) we have

$$(2.24) \quad b_{k+p} = 4^p b_k + \sum_{r=1}^p d_{k+r} 4^{p-r} = 4^p b_k + b'_k.$$

We first compute  $a(b_{k+p})$  using (1.4), in which  $e(n)$  is the number of pairs of consecutive ones in the binary representation of  $n$ . Now the binary representation of  $b_{k+p}$  is pieced together from the binary representations of  $b_k$  and  $b'_k$ , by (2.24). Moreover, a 1 occurs simultaneously in the last binary digit of  $b_k$  and the first binary digit of  $b'_k$  if and only if  $2 \nmid d_k$  and  $d_{k+1} = 2$  or  $3$ . Thus we have

$$\begin{aligned} a(b_{k+p}) &= a(b_k)a(b'_k)(-1)^{d_k[d_{k+1}/2]} \\ &= a(b_k)\varepsilon_k, \quad \text{for } k \geq k_0, \end{aligned}$$

where  $\varepsilon_k = \pm 1$ . Since  $b'_{k+p} = b'_k$ ,  $\varepsilon_{k+p} = \varepsilon_k$  for  $k \geq k_0$ ; we deduce that

$$a(b_{k+2p}) = a(b_{k+p})\varepsilon_k\varepsilon_{k+p} = a(b_k),$$

and this proves (2.23). □

**COROLLARY.** *If  $x > 0$  and  $x \in \mathbf{Q}$ , then  $\sqrt{x}\mu(x) \in \mathbf{Q}$ .*

*Proof.* This is clear from (2.19).

As a further example of Theorem 3 we note that

$$\frac{1}{\sqrt{73}}\lambda\left(\frac{1}{73}\right) = \frac{65297}{65408} = \frac{17 \cdot 23 \cdot 167}{2^7 \cdot 7 \cdot 73},$$

where the value

$$a\left(\frac{1}{73}\right) = \frac{111}{65408} = \frac{3 \cdot 37}{2^7 \cdot 7 \cdot 73}$$

is readily obtained from 4-adic expansion

$$\frac{1}{73} = \overline{.000320013}.$$

We remark that the converse of Theorem 3 is certainly false, since there are irrational numbers  $x = \sum_{r=0}^{\infty} d_r 4^{-r}$  for which  $d_r$  is always odd; for these  $x$  we have  $a(x) = 0$  from (2.3), so  $\sqrt{x} \lambda(x) = s(x) \in \mathbb{Z}$ .

**3. The continuity of  $\lambda(x)$  and  $\mu(x)$ .** In this section we show that  $\lambda(x)$  and  $\mu(x)$  are actually continuous functions of  $x$ , for  $x > 0$ . Equation (2.19) shows that it is enough to prove this for  $\lambda(x)$ .

We first consider the function  $a(x)$ .

**THEOREM 4.** *Let  $x_0 > 0$ . Then  $a(x)$  is continuous at  $x_0$  if and only if  $x_0$  is not a natural number. If  $x_0$  is a natural number, then*

$$(3.1) \quad \lim_{x \rightarrow x_0^-} a(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow x_0^+} a(x) = a(x_0) = \pm 1.$$

*Proof.* We prove the theorem in three parts:

- (i)  $a(x)$  is continuous from the right at any  $x_0 > 0$ ;
- (ii)  $a(x)$  is continuous from the left at  $x_0 \notin \mathbb{N}$ ;
- (iii)  $\lim_{x \rightarrow x_0^-} a(x) = 0$ , if  $x_0 \in \mathbb{N}$ .

Here  $\mathbb{N}$  denotes the set of natural numbers.

(i) Assume  $x_0 = \sum_{r=0}^{\infty} d_r 4^{-r}$  as in (2.5), and define  $x_n$  by  $4^n x_n = [4^n x_0] + 1 = b_n + 1$ , for  $n \geq 1$ , so that  $x_n > x_0$  and  $x_n \rightarrow x_0$  as  $n \rightarrow \infty$ . If  $x_0 < x^* < x_n$ , then  $x^* = \sum_{r=0}^{\infty} d_r^* 4^{-r}$  with  $d_r^* = d_r$  for  $0 \leq r \leq n$ . Hence, by (2.6),  $b_r^* = [4^r x^*] = [4^r x_0] = b_r$  for  $0 \leq r \leq n$ , and by (2.12) we have that

$$\begin{aligned} |a(x_0) - a(x^*)| &= \left| \sum_{r=1}^{\infty} \rho(d_r) a(b_r) 2^{-r} - \sum_{r=1}^{\infty} \rho(d_r^*) a(b_r^*) 2^{-r} \right| \\ &\leq \sum_{r=n+1}^{\infty} |\rho(d_r) a(b_r) - \rho(d_r^*) a(b_r^*)| 2^{-r} \leq \sum_{r=n+1}^{\infty} 2^{1-r} = 2^{1-n}. \end{aligned}$$

This clearly implies (i).

(ii) Here there are two cases:

(a) If  $x_0 = \sum_{r=0}^{\infty} d_r 4^{-r}$ , where infinitely many  $d_r$  are nonzero, then we set  $x_n = \sum_{r=0}^n d_r 4^{-r}$ , so that  $x_n < x_0$  and  $x_n \rightarrow x_0$  as  $n \rightarrow \infty$ . If  $x^*$  satisfies  $x_n < x^* < x_0$ , then clearly  $x^* = \sum_{r=0}^{\infty} d_r^* 4^{-r}$  with  $d_r^* = d_r$  for  $0 \leq r \leq n$ , and as in (i) we find that  $|a(x_0) - a(x^*)| \leq 2^{1-n}$ .

(b) In the second case,  $x_0 = \sum_{r=0}^s d_r 4^{-r}$ , where  $s \geq 1$  and  $d_s \neq 0$ . Let  $n \geq s + 1$  and define

$$x_n = x_0 - 4^{-n} = \sum_{r=0}^{s-1} d_r 4^{-r} + \frac{d_s - 1}{4^s} + \sum_{r=s+1}^n 3 \cdot 4^{-r}.$$

For any  $x^*$  in the interval  $x_n < x^* < x_0$ , we then have  $x^* = \sum_{r=0}^{\infty} d_r^* 4^{-r}$ , with

$$d_r^* = \begin{cases} d_r, & \text{for } 0 \leq r \leq s-1, \\ d_s - 1, & \text{for } r = s, \\ 3, & \text{for } s+1 \leq r \leq n. \end{cases}$$

Thus, we see from (2.12) that

$$\begin{aligned} a(x^*) &= \sum_{r=1}^n \rho(d_r^*) a(b_r^*) 2^{-r} + \sum_{r=n+1}^{\infty} \rho(d_r^*) a(b_r^*) 2^{-r} \\ &= \sum_{r=1}^{s-1} \rho(d_r) a(b_r) 2^{-r} + \rho(d_s - 1) a(b_s^*) 2^{-s} + \sum_{r=n+1}^{\infty} \rho(d_r^*) a(b_r^*) 2^{-r} \\ &= \sum_{r=1}^{s-1} \rho(d_r) a(b_r) 2^{-r} + \rho(d_s - 1) a(b_s - 1) 2^{-s} + O(2^{-n}), \end{aligned}$$

since  $b_s^* = 4b_{s-1}^* + d_s^* = 4b_{s-1} + d_s - 1 = b_s - 1$ . On the other hand,

$$\begin{aligned} a(x_0) &= \sum_{r=1}^s \rho(d_r) a(b_r) 2^{-r} + \sum_{r=s+1}^{\infty} \rho(0) a(b_r) 2^{-r} \\ &= \sum_{r=1}^{s-1} \rho(d_r) a(b_r) 2^{-r} + \rho(d_s) a(b_s) 2^{-s} + a(b_s) \sum_{r=s+1}^{\infty} 2^{-r} \\ &= \sum_{r=1}^{s-1} \rho(d_r) a(b_r) 2^{-r} + \rho(d_s) a(b_s) 2^{-s} + a(b_s) 2^{-s}, \end{aligned}$$

and subtracting the expressions for  $a(x^*)$  and  $a(x_0)$  gives

$$\begin{aligned} a(x^*) - a(x_0) &= [\rho(d_s - 1) a(b_s - 1) - \rho(d_s) a(b_s) - a(b_s)] 2^{-s} + O(2^{-n}). \end{aligned}$$

We now claim that the expression  $E_s$  inside the brackets is zero. To show this we must consider the three possibilities:  $d_s = 1, 2$ , or  $3$  (note  $d_s \neq 0$  by assumption). Recall that  $b_s = 4b_{s-1} + d_s$ .

If  $d_s = 1$ , then

$$\begin{aligned} E_s &= \rho(0) a(b_s - 1) - a(b_s) = a(4b_{s-1}) - a(4b_{s-1} + 1) \\ &= a(b_{s-1}) - a(b_{s-1}) = 0, \quad \text{by (1.3).} \end{aligned}$$

If  $d_s = 2$ , then  $E_s = -\rho(2)a(b_s) - a(b_s) = 0$ .

If  $d_s = 3$ , then  $E_s = \rho(2)a(b_s - 1) - a(b_s) = -a(4b_{s-1} + 2) - a(4b_{s-1} + 3) = -a(2b_{s-1} + 1) + a(2b_{s-1} + 1) = 0$ , again by (1.3).

Thus, we have that  $|a(x^*) - a(x_0)| = O(2^{-n})$ , when  $x_n < x^* < x_0$ , for any  $n \geq s + 1$ , and this shows that  $a(x)$  is continuous from the left at  $x_0$ .

(iii) Assume now that  $x_0 \in \mathbf{N}$ , and define

$$x_n = x_0 - 4^{-n} = x_0 - 1 + \sum_{r=1}^n 3 \cdot 4^{-r}, \quad n \geq 1.$$

As in (ii) we have for any  $x^*$  in the interval  $x_n < x^* < x_0$  that  $x^* = \sum_{r=0}^{\infty} d_r^* 4^{-r}$ , where

$$d_r^* = \begin{cases} x_0 - 1, & \text{for } r = 0, \\ 3, & \text{for } 1 \leq r \leq n. \end{cases}$$

Hence,  $a(x^*) = \sum_{r=1}^n \rho(3)a(b_r^*)2^{-r} + O(2^{-n}) = O(2^{-n})$ , since  $\rho(3) = 0$ . But this implies  $a(x^*) \rightarrow 0$  as  $x^* \rightarrow x_0$  from below.  $\square$

REMARK. The same proof shows that the complex valued function

$$(3.2) \quad a_{\tau}(x) = \sum_{r=1}^{\infty} \rho(d_r)a(b_r)2^{-\tau r},$$

defined for complex numbers  $\tau$  with positive real part, is continuous at  $x_0$  whenever  $x_0 \notin \mathbf{N}$ , and that

$$\lim_{x \rightarrow x_0^-} a_{\tau}(x) = 0, \quad \lim_{x \rightarrow x_0^+} a_{\tau}(x) = a_{\tau}(x_0), \quad \text{if } x_0 \in \mathbf{N}.$$

THEOREM 5.  $\lambda(x)$  is continuous for  $x > 0$ .

*Proof.* Let  $x_0 > 0$ . If  $x_0 \notin \mathbf{N}$ , then it follows from Theorem 4, equation (2.13), and the fact that  $s(x)$  is a step-function that  $\lambda(x)$  is continuous at  $x_0$ . If  $x_0 \in \mathbf{N}$ , the same considerations show that  $\lambda(x)$  is continuous from the right at  $x_0$ . Furthermore, by (2.13), (3.1), and (2.10) we have that

$$\begin{aligned} \lim_{x^* \rightarrow x_0^-} \lambda(x^*) &= \lim_{x^* \rightarrow x_0^-} [s(x_0 - 1) - a(x^*)](x^*)^{-1/2} \\ &= \frac{s(x_0 - 1)}{\sqrt{x_0}} = \lambda(x_0). \end{aligned}$$

Therefore  $\lambda(x)$  is continuous at  $x_0$ .  $\square$

**COROLLARY 1.** *The function  $\lambda(x)$  maps both intervals  $(0, \infty)$  and  $[1, 4]$  continuously onto  $[\sqrt{3/5}, \sqrt{6}]$ .*

*Proof.* This is immediate from Theorem 5, (2.14) and Lemma 2. Alternatively, one may deduce Corollary 1 from the intermediate value theorem and the values  $\lambda(5/3) = \sqrt{3/5}$ ,  $\lambda(8/3) = \sqrt{6}$ .

**COROLLARY 2.** *The function  $\mu(x)$  maps  $(0, \infty)$  and  $[1, 4]$  continuously onto  $[0, \sqrt{3}]$ .*

We remark that the continuity of  $\lambda(x)$  for  $x > 0$  also follows from the fact that the functions  $f_k(x) = s(4^k x)(4^k x)^{-1/2}$  converge uniformly to  $\lambda(x)$  on any interval  $[a, b]$  with  $0 < a < b$ , by (2.15). The functions  $f_k(x)$  are step functions with jump discontinuities of order  $2^{-k}x^{-1/2}$  at the points  $x$  for which  $4^k x \in \mathbb{N}$ . The continuity of  $\lambda(x)$  may then be deduced from the following general result, whose proof we leave to the reader.

**THEOREM.** *Let  $J$  be an interval, and let  $\{f_k(x)\}$  be a sequence of functions converging uniformly to  $f(x)$  on  $J$ . Assume for every  $x_0$  in  $J$  that*

$$d_k(x_0) = \limsup_{x \rightarrow x_0} |f_k(x) - f_k(x_0)| \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

*Then  $f(x)$  is continuous on  $J$ .*

**4. The non-differentiability of  $\lambda(x)$ .** Although  $\lambda(x)$  is a continuous function, it is differentiable almost nowhere. To prove this we first recall the following definition. (See [6], Ch. 8.)

**DEFINITION.** A real number  $x_0 > 0$  is normal (to the base 4) if and only if the numbers  $x_0, 4x_0, 4^2x_0, \dots, 4^n x_0, \dots$  are uniformly distributed modulo 1.

An equivalent definition is the following. Let  $k \geq 1$ , and let  $B_k$  be a block of  $k$  digits to the base 4. Also let  $x_0 = \sum_{r=0}^{\infty} d_r 4^{-r}$ , and denote by  $N(m, B_k)$  the number of occurrences of the block  $B_k$  in the initial block  $.d_1 d_2 \dots d_m$  of  $x_0 - d_0$ . (For example, if  $x_0 = .1121121102$  and  $B_5 = 11211$ , we have  $N(10, B_5) = 2$ .) Then  $x_0$  is normal if and only if

$$(4.1) \quad \lim_{m \rightarrow \infty} \frac{1}{m} N(m, B_k) = 4^{-k},$$

for all  $k \geq 1$  and all blocks  $B_k$  of length  $k$ .

It is well-known [6] that almost all positive real numbers are normal. In particular, almost all positive real numbers  $x_0 = \sum_{r=0}^{\infty} d_r 4^{-r}$  have the property that  $d_n = d_{n+1} = 0$  for infinitely many  $n$ . This is the essential fact we use in proving

**THEOREM 6.** *If  $x_0 > 0$  is normal (to the base 4), then  $\lambda(x)$  is not differentiable at  $x_0$ . Thus,  $\lambda(x)$  is non-differentiable almost everywhere.*

*Proof.* Since  $\sqrt{x}\lambda(x) = s(x) - a(x)$ , it is enough to prove that

$$(4.2) \quad \frac{1}{h} \{a(x_0 + h) - a(x_0)\}$$

is unbounded as  $h \rightarrow 0^+$ . The theorem then follows from the fact that the step function  $s(x)$  has right derivative 0 for all  $x_0 > 0$ .

So let  $x_0 = \sum_{r=0}^{\infty} d_r 4^{-r}$ , choose an  $n \geq 1$  for which  $d_n = d_{n+1} = 0$ , and set  $h = 4^{-n}$ . Then the 4-adic expansion of  $x_0 + h$  is

$$x_0 + h = \sum_{r=0}^{n-1} d_r 4^{-r} + 4^{-n} + \sum_{r=n+1}^{\infty} d_r 4^{-r}.$$

Putting  $b'_r = [4^r(x_0 + 4^{-n})]$ , we have  $b'_r = b_r$  for  $r \leq n-1$ , while  $b'_n = 4b_{n-1} + 1 = b_n + 1$  and  $b'_{n+1} = 4b'_n = b_{n+1} + 4$ . Thus (1.3) implies  $a(b'_n) = a(b_{n-1}) = a(b_n)$  and  $a(b'_{n+1}) = a(b'_n) = a(b_n) = a(b_{n+1})$ . Furthermore, using (1.4) and considering the binary expansions of  $b'_m$  and  $b_m$ , we see that  $a(b'_m) = a(b_m)$ , for  $m \geq n+2$ . Hence

$$\begin{aligned} a(x_0 + 4^{-n}) - a(x_0) &= \sum_{r=1}^{n-1} \frac{\rho(d_r)a(b_r)}{2^r} + \sum_{r=n+1}^{\infty} \frac{\rho(d_r)a(b'_r)}{2^r} \\ &\quad - \sum_{r=1}^{\infty} \frac{\rho(d_r)a(b_r)}{2^r} = -a(b_n)2^{-n}, \end{aligned}$$

and so

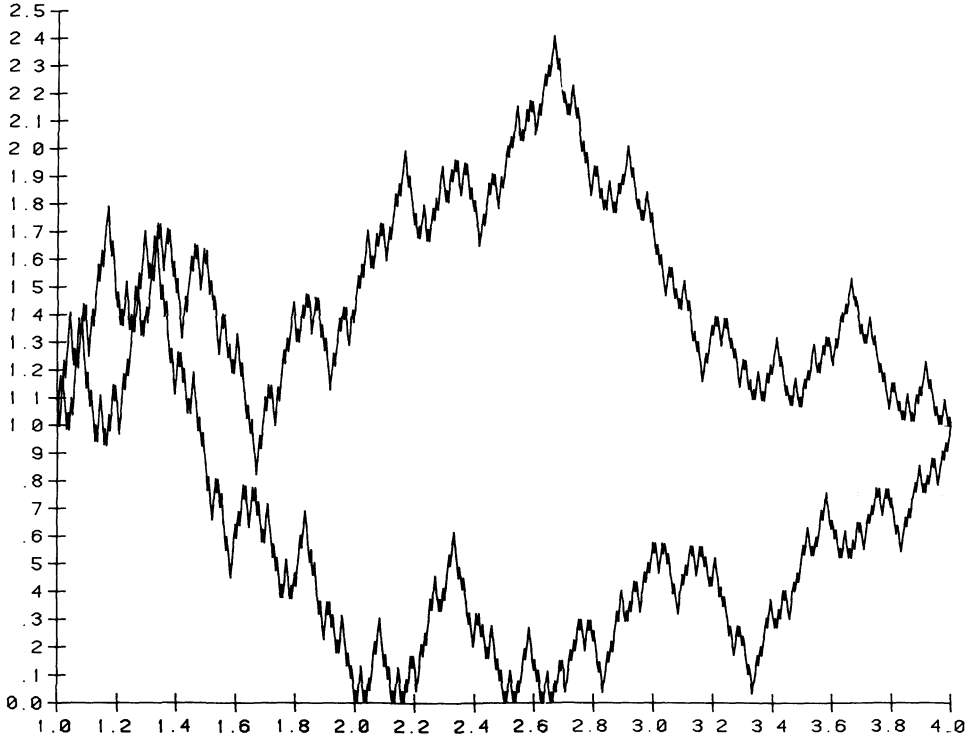
$$(4.3) \quad 4^n \{a(x_0 + 4^{-n}) - a(x_0)\} = -a(b_n)2^n = \pm 2^n.$$

Since there are infinitely many  $n$  for which  $d_n = d_{n+1} = 0$ , this proves that the expression (4.2) is indeed unbounded as  $h \rightarrow 0^+$ .  $\square$

We remark that the same proof shows  $\lambda(x)$  is not differentiable at any positive rational  $x_0$  whose denominator is a power of 2.

The proof of Theorem 6 can also be modified to show that for a normal number  $x_0$ , the quotient (4.2) takes on all real values infinitely often as  $h \rightarrow 0^+$ . For one can choose a sequence  $n_k$  with  $d_{n_k} = d_{n_k+1} = 0$ ,

$k \geq 1$ , such that  $a(b_{n_k})$  changes sign infinitely often. (For example, the block 00300 occurs infinitely often among the digits of  $x_0$ . If the block starts at the index  $n$ , and  $n_k = n$ ,  $n_{k+1} = n + 3$ , then  $a(b_{n_{k+1}}) = -a(b_{n_k})$ .) It follows from (4.3) that the quotient (4.2), which is continuous in  $h$  for small  $h$ , takes on arbitrarily large positive and negative values as  $h \rightarrow 0^+$ . The intermediate value theorem then shows the truth of the claim above. This remark is due to A. J. E. M. Janssen (private communication).



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FIGURE 1. Polygonal approximations to  $\lambda(x)$  and  $\mu(x)$ , ( $\lambda(x) \geq \mu(x)$ .)

The upper graph in Figure 1 is the polygonal curve joining the points

$$\left(1 + \frac{n}{4^5}, \lambda\left(1 + \frac{n}{4^5}\right)\right) = \left(1 + \frac{n}{4^5}, \frac{s(4^5 + n - 1)}{\sqrt{4^5 + n}}\right), \quad n = 0, 1, \dots, 3 \cdot 4^5.$$

The lower graph is the same with the function  $s$  replaced by the function  $t$ , and  $\lambda$  replaced by  $\mu$ .

**5. The Fourier series of  $\lambda(x)$ .** It follows from the continuity of  $\lambda(x)$  and (2.14) that the function

$$(5.1) \quad f(x) = \lambda(4^{x/2\pi})$$

is continuous for all  $x$  and has period  $2\pi$ . Thus  $f$  has a Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad c_n = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-in\theta} d\theta,$$

which is  $(C, 1)$  summable to  $f(x)$  for all  $x$ . (See [3], p. 62) Using  $\lambda(x) = f(\pi \log x / \log 2)$ , this easily yields the following result for  $\lambda(x)$ .

**THEOREM 7.** *The function  $\lambda(x)$  has the logarithmic Fourier series expansion*

$$(5.2) \quad \lambda(x) = \sum_{n=-\infty}^{\infty} c_n e^{\pi i n \log x / \log 2}, \quad x > 0,$$

where

$$(5.3) \quad c_n = \frac{1}{\log 4} \int_1^4 \frac{\lambda(x)}{x^{1/2+\gamma_n}} dx, \quad \gamma_n = \frac{1}{2} + \frac{\pi n i}{\log 2},$$

and where the infinite series converges in the  $(C, 1)$  sense for all  $x > 0$ , i.e.

$$\sum_{n=-\infty}^{\infty} c_n e^{\pi i n \log x / \log 2} = \lim_{k \rightarrow \infty} \frac{1}{k+1} (\sigma_0 + \sigma_1 + \cdots + \sigma_k),$$

with

$$\sigma_k = \sum_{n=-k}^k c_n e^{\pi i n \log x / \log 2}.$$

**COROLLARY.** *For  $x > 0$  we have*

$$(5.4) \quad s(x) = \sum_{n=-\infty}^{\infty} c_n x^{1/2+\pi n i / \log 2} + a(x),$$

where the series is  $(C, 1)$  summable for all  $x > 0$ , and  $c_n$  is defined by (5.3).

*Proof.* This is immediate from (5.2) and (2.13).

We note that the series in (5.2) and (5.4) are convergent in the usual sense for almost all  $x > 0$ , by the deep theorem of Carleson [2]. However,



it is possible to give a direct proof of this fact. We first prove

LEMMA 5. *If  $x_0 > 0$  is normal (to the base 4), then*

$$(5.5) \quad |a(x_0 + h) - a(x_0)| = O(|h|^{1/4}), \quad \text{as } h \rightarrow 0,$$

where the implied constant depends only on  $x_0$ .

*Proof.* Let  $x_0 = \sum_{r=0}^{\infty} d_r 4^{-r}$ , and assume  $4^{-n-1} \leq h < 4^{-n}$ ,  $n \geq 1$ , so that

$$h = \sum_{r=n+1}^{\infty} h_r 4^{-r}, \quad 0 \leq h_r \leq 3, h_{n+1} \neq 0.$$

Then

$$x_0 + h = \sum_{r=0}^n d_r 4^{-r} + \sum_{r=n+1}^{\infty} (d_r + h_r) 4^{-r}.$$

Since  $d_r$  and  $h_r$  are digits, not all equal to 3 past some point, we have that

$$\sum_{r=n+1}^{\infty} (d_r + h_r) 4^{-r} < \sum_{r=n+1}^{\infty} 6 \cdot 4^{-r} = 2 \cdot 4^{-n},$$

so

$$(5.6) \quad x_0 + h = \sum_{r=0}^n d_r 4^{-r} + \sum_{r=n}^{\infty} h'_r 4^{-r},$$

where the  $h'_r$  are digits and  $h'_n = 0$  or 1. If  $h'_n = 1$ , then there is a carry into the  $n$ th place in (5.6). However the carrying will stop as soon as some  $d_r \neq 3$ ,  $r \leq n$ .

In order to estimate how long the carrying continues, we apply (4.1) to the number  $x_0$  and the block  $B_1 = 3$ . By that equation we may choose an  $n_0$  so that

$$N(m, B_1) < \frac{3m}{8}, \quad \text{for } m \geq n_0.$$

Therefore, if  $n \geq n_0$ , the number of digits  $d_r$  equal to 3 between  $n/2$  and  $n$  is at most  $3n/8 < n/2$ . Hence there is an  $r_0 > n/2$  for which  $d_{r_0} \neq 3$ , and this implies that

$$x_0 + h = \sum_{r=0}^{r_0-1} d_r 4^{-r} + \sum_{r \geq r_0} d'_r 4^{-r},$$

where the  $d'_r$  are digits.

Now apply (2.12) with  $b'_r = [4^r x_0 + 4^r h]$ ,  $b_r = [4^r x_0]$ , to give

$$\begin{aligned} |a(x_0 + h) - a(x_0)| &= \left| \sum_{r \geq r_0} \rho(d'_r) a(b'_r) 2^{-r} - \sum_{r \geq r_0} \rho(d_r) a(b_r) 2^{-r} \right| \\ &\leq 2 \sum_{r \geq r_0} 2^{-r} = \frac{4}{2^{r_0}} < \frac{4}{2^{n/2}} \leq 4\sqrt{2} h^{1/4}, \end{aligned}$$

for  $n \geq n_0$ . Thus

$$|a(x_0 + h) - a(x_0)| = O(h^{1/4}), \quad \text{as } h \rightarrow 0^+.$$

A similar discussion shows that

$$|a(x_0 - h) - a(x_0)| = O(h^{1/4}), \quad \text{as } h \rightarrow 0^+,$$

and this completes the proof of the lemma.

**THEOREM 8.** *If  $x_0 > 0$  is a normal number (to the base 4), then the Fourier series (5.2) of  $\lambda(x)$  converges at  $x_0$ . Thus, (5.2) and (5.4) converge for almost all positive real numbers  $x$ .*

*Proof.* Since  $x_0$  is not an integer,  $s(x_0 + h) = s(x_0)$  for small  $h$ , and so (2.13) gives that

$$\begin{aligned} \lambda(x_0 + h) - \lambda(x_0) &= \frac{s(x_0)}{\sqrt{x_0 + h}} - \frac{s(x_0)}{\sqrt{x_0}} - \frac{a(x_0 + h)}{\sqrt{x_0 + h}} + \frac{a(x_0)}{\sqrt{x_0}} \\ &= \frac{s(x_0)}{\sqrt{x_0 + h}} \left( 1 - \sqrt{1 + \frac{h}{x_0}} \right) - \frac{a(x_0 + h) - a(x_0)}{\sqrt{x_0 + h}} \\ &\quad + \frac{a(x_0)}{\sqrt{x_0 + h}} \left( \sqrt{1 + \frac{h}{x_0}} - 1 \right). \end{aligned}$$

Now  $(x_0 + h)^{-1/2}$  is bounded as  $h \rightarrow 0$ , and  $(1 + h/x_0)^{1/2} = 1 + h/2x_0 + O(h^2) = 1 + O(|h|^{1/4})$ , as  $h \rightarrow 0$ . Therefore, Lemma 5 implies that

$$|\lambda(x_0 + h) - \lambda(x_0)| = O(|h|^{1/4}), \quad \text{as } h \rightarrow 0.$$

We set  $y = 1 + h/x_0$ , and use the fact that  $h \simeq x_0 \log y$  as  $h \rightarrow 0$  to write the last estimate in the form

$$|\lambda(x_0 y) - \lambda(x_0)| = O(|\log y|^{1/4}), \quad \text{as } y \rightarrow 1.$$

If  $z_0 = \pi \log x_0 / \log 2$ , then this gives the following estimate for the function  $f(z) = \lambda(4^{z/2\pi})$ :

$$|f(z_0 + h) - f(z_0)| = O(|h|^{1/4}), \quad \text{as } h \rightarrow 0.$$

But this condition implies the convergence of the Fourier series of  $f$  at  $z_0$  (see [3], p. 41), and therefore the convergence of (5.2) at  $x_0$ .  $\square$

REMARK. If we define a *simply normal* number to be a number  $x_0$  which satisfies the condition (4.1) just for  $k = 1$ , i.e. for blocks of length one, then it is clear that the conclusions of Lemma 5 and Theorem 8 hold for the larger set of simply normal numbers. Thus both (5.2) and (5.4) converge for example at the point

$$x = m.01230123 \cdots_4 = m + \frac{9}{85},$$

where  $m$  is a non-negative integer. Similarly, (5.2) and (5.4) converge at any point  $x_0 = \sum_{r=0}^{\infty} d_r 4^{-r}$  which has the property that  $d_r \neq 0$  or 3 for large  $r$ , e.g. the point  $x_0 = .1212 \cdots_4 = 2/5$ .

Our results for  $\lambda(x)$  and  $s(x)$  are easily extended to the functions  $\mu(x)$  and  $t(x)$  using (2.18) and (2.19). For example,  $\mu(x)$  has the logarithmic Fourier series

$$\mu(x) = \sum_{n=-\infty}^{\infty} c_n \{(-1)^n \sqrt{2} - 1\} e^{\pi i n \log x / \log 2},$$

which is  $(C, 1)$  summable to  $\mu(x)$  for all  $x > 0$ , and which is actually convergent in case  $x$  is normal to the base 4 (for then  $2x$  is also normal). Moreover, (2.18) implies easily that

$$\begin{aligned} t(x) &= s(2x) - s(x) + \frac{1}{2} (1 + (-1)^{[d_1/2]}) (-1)^{d_0} a(b_0) \\ &= \sqrt{x} \mu(x) + b(x) \\ &= \sum_{n=-\infty}^{\infty} c_n \{(-1)^n \sqrt{2} - 1\} x^{1/2 + \pi n i / \log 2} + b(x), \end{aligned}$$

where

$$b(x) = a(2x) - a(x) + \frac{1}{2} (1 + (-1)^{[d_1/2]}) (-1)^{d_0} a(b_0),$$

and  $x$  is given by (2.5). The function  $b(x)$  has properties analogous to those of  $a(x)$ . For instance,  $b(n) = (-1)^n a(n)$ , for  $n \geq 0$ ;  $b(x)$  is continuous at  $x_0$  if  $x_0 \neq \mathbf{N}$ ; and

$$\lim_{x \rightarrow x_0^-} b(x) = 0, \quad \lim_{x \rightarrow x_0^+} b(x) = b(x_0), \quad \text{if } x_0 \in \mathbf{N}.$$

**6. The Fourier coefficients  $c_n$ .** Concerning the coefficients  $c_n$ , we first prove

**THEOREM 9.** *Infinitely many of the coefficients  $c_n$  are nonzero; in fact  $c_n \neq O(|n|^{-2-\delta})$ , as  $n \rightarrow \pm \infty$ , for any  $\delta > 0$ .*

*Proof.* Assume that  $c_n = O(|n|^{-2-\delta})$  for some  $\delta > 0$ . Then the series in (5.2) converges to  $\lambda(x)$  for all  $x > 0$ , and the differentiated series

$$\frac{\pi i}{x \log 2} \sum_{n=-\infty}^{\infty} n c_n x^{\pi i n / \log 2} = \Delta(x)$$

converges uniformly for  $x \geq 1$ . Therefore  $\lambda'(x) = \Delta(x)$  for all  $x > 1$ , which contradicts Theorem 6. Hence  $c_n = O(|n|^{-2-\delta})$  is false.  $\square$

We shall now relate the  $c_n$  to the behavior of the function  $\eta(\tau)$  defined by the Dirichlet series

$$(6.1) \quad \eta(\tau) = \sum_{n=1}^{\infty} \frac{a(n)}{n^{\tau}}.$$

By virtue of (1.5), this series converges in the half-plane  $\operatorname{Re} \tau > 1/2$ , and absolutely for  $\operatorname{Re} \tau > 1$ . (See [5], p. 123.) Using partial summation to express  $\eta(\tau)$  as an integral gives

$$\begin{aligned} \eta(\tau) &= \sum_{n=1}^{\infty} \frac{s(n) - s(n-1)}{n^{\tau}} = -1 + \sum_{n=1}^{\infty} s(n) \left\{ \frac{1}{n^{\tau}} - \frac{1}{(n+1)^{\tau}} \right\} \\ &= -1 + \tau \sum_{n=1}^{\infty} s(n) \int_n^{n+1} \frac{1}{x^{\tau+1}} dx \\ &= -1 + \tau \int_1^{\infty} \frac{s(x)}{x^{\tau+1}} dx, \quad \text{for } \operatorname{Re} \tau > \frac{1}{2}. \end{aligned}$$

We substitute  $s(x) = \sqrt{x} \lambda(x) + a(x)$ , and find

$$(6.2) \quad \eta(\tau) = -1 + \tau \int_1^{\infty} \frac{\lambda(x)}{x^{\tau+1/2}} dx + \tau \int_1^{\infty} \frac{a(x)}{x^{\tau+1}} dx.$$

Now rearrange the first integral using (2.14):

$$\begin{aligned}
 (6.3) \quad \int_1^\infty \frac{\lambda(x)}{x^{\tau+1/2}} dx &= \sum_{k=0}^\infty \int_{4^k}^{4^{k+1}} \frac{\lambda(x)}{x^{\tau+1/2}} dx \\
 &= \sum_{k=0}^\infty \int_1^4 \frac{\lambda(x)}{2^{k(2\tau-1)} u^{\tau+1/2}} du, \\
 &= (1 - 2^{1-2\tau})^{-1} \int_1^4 \frac{\lambda(x)}{x^{\tau+1/2}} dx, \quad \operatorname{Re} \tau > \frac{1}{2}.
 \end{aligned}$$

Similarly, the second integral may be written in the form

$$\begin{aligned}
 (6.4) \quad \int_1^\infty \frac{a(x)}{x^{\tau+1}} dx &= \sum_{k=0}^\infty \int_{4^k}^{4^{k+1}} \frac{a(x)}{x^{\tau+1}} dx \\
 &= \sum_{k=0}^\infty \int_1^4 \frac{a(4^k x)}{2^{2k\tau} x^{\tau+1}} dx \\
 &= \int_1^4 \frac{1}{x^{\tau+1}} \sum_{k=0}^\infty \frac{a(4^k x)}{2^{2k\tau}} dx.
 \end{aligned}$$

To evaluate the integrand, we need the following result.

LEMMA 6. *In the notation of (2.5) and (2.6) we have that*

$$(6.5) \quad a(4^k x) = 2^k a(x) - \sum_{r=1}^k \rho(d_r) a(b_r) 2^{k-r}, \quad \text{for } x > 0, k \geq 1.$$

*Proof.* From (2.14) we have  $\lambda(4^k x) = \lambda(x)$ , so from (2.13) we find that

$$s(4^k x) - 2^k s(x) = a(4^k x) - 2^k a(x).$$

Equation (6.5) is now immediate from (2.9).

With (6.5) we can write the infinite sum in (6.4) as follows:

$$\begin{aligned}
 \sum_{k=0}^\infty \frac{a(4^k x)}{2^{2k\tau}} &= \sum_{k=0}^\infty \frac{a(x)}{2^{k(2\tau-1)}} - \sum_{k=1}^\infty 2^{-2k\tau} \sum_{r=1}^\infty \rho(d_r) a(b_r) 2^{k-r} \\
 &= (1 - 2^{1-2\tau})^{-1} a(x) - \sum_{r=1}^\infty \rho(d_r) a(b_r) 2^{-r} \sum_{k=r}^\infty 2^{k(1-2\tau)} \\
 &= (1 - 2^{1-2\tau})^{-1} a(x) - (1 - 2^{1-2\tau})^{-1} a_{2\tau}(x),
 \end{aligned}$$

where  $a_\tau(x)$  is defined by (3.2).

Putting the results of (6.6), (6.4), and (6.3) into (6.2) gives finally that

$$(6.7) \quad (1 - 2^{1-2\tau})\eta(\tau) = 2^{1-2\tau} - 1 + \tau \int_1^4 \frac{\lambda(x)}{x^{\tau+1/2}} dx + \tau \int_1^4 \frac{a(x) - a_{2\tau}(x)}{x^{\tau+1}} dx,$$

initially for  $\operatorname{Re} \tau > 1/2$ . But the integrals in this formula define analytic functions of  $\tau$  for  $\operatorname{Re} \tau > 0$ . (In fact the first integral is entire.) Thus (6.7) defines the analytic continuation of  $\eta(\tau)$  to the half-plane  $\operatorname{Re} \tau > 0$ , and  $\eta$  has at most simple poles at the points  $\tau$  for which  $2^{1-2\tau} = 1$ , i.e. the points  $\gamma_n = 1/2 + \pi ni/\log 2$ ,  $n \in \mathbf{Z}$ . This proves

**THEOREM 10.** *The function  $\eta(\tau)$  defined by (6.1) has a meromorphic continuation to the half-plane  $\operatorname{Re} \tau > 0$ , with at most simple poles at the points  $\gamma_n = 1/2 + \pi ni/\log 2$ ,  $n \in \mathbf{Z}$ .*

In fact, the function  $\eta(\tau)$  has a meromorphic continuation to the whole complex plane, but we shall not give the proof of this fact here. Rather, we point out the following connection between  $c_n$  and the behavior of  $\eta(\tau)$  at the point  $\tau = \gamma_n$ .

**THEOREM 11.** *The  $n$ th Fourier coefficient  $c_n$  of  $\lambda(x)$  is related to the residue  $R_n$  of  $\eta(\tau)$  at  $\gamma_n$  by the formula*

$$(6.8) \quad c_n = R_n/\gamma_n = \eta_0(\gamma_n)/(\gamma_n \log 4),$$

where  $\eta_0(\tau) = (1 - 2^{1-2\tau})\eta(\tau)$ .

*Proof.* Since  $2^{2\gamma_n} = 2$ , we have  $a_{2\gamma_n}(x) = a(x)$  for all  $n \in \mathbf{Z}$  and  $x > 0$ . Putting  $\tau = \gamma_n$  in (6.7) gives therefore that

$$\eta_0(\gamma_n) = \gamma_n \int_1^4 \frac{\lambda(x)}{x^{1/2+\gamma_n}} dx = \gamma_n \log 4 \cdot c_n.$$

Equation (6.8) is immediate from this and the fact that  $\eta_0(\gamma_n) = \log 4 \cdot R_n$ .

**COROLLARY 1.** *Infinitely many of the points  $\gamma_n$  are simple poles of  $\eta(\tau)$ . In fact,  $R_n \neq O(|n|^{-1-\delta})$ , as  $n \rightarrow \pm \infty$ , for any  $\delta > 0$ .*

*Proof.* Immediate from (6.8) and Theorem 9.

Equation (6.8) can also be used to estimate the size of  $c_n$ . To do this we note the Dirichlet series expansion for  $\eta_0(\tau)$ :

$$(6.9) \quad \eta_0(\tau) = (1 - 2^{1-2\tau})\eta(\tau) = \sum_{n=1}^{\infty} \frac{a(n)}{n^{\tau}} - 2 \sum_{n=1}^{\infty} \frac{a(n)}{(4n)^{\tau}} \\ = \sum_{n=1}^{\infty} \frac{a^*(n)}{n^{\tau}},$$

where

$$(6.10) \quad a^*(n) = \begin{cases} a(n), & \text{if } 4 \nmid n, \\ -a(n), & \text{if } 4 \mid n. \end{cases}$$

If we set

$$s^*(x) = \sum_{k=0}^{\lfloor x \rfloor} a^*(k),$$

in analogy to (1.1), then it is easy to see that

$$s^*(x) = s(x) - 2s(x/4) = a(x) - 2a(x/4) \\ = O(1), \quad \text{as } x \rightarrow \infty.$$

Hence (6.9) converges for  $\text{Re } \tau > 0$ . This implies the following corollary to Theorem 11.

**COROLLARY 2.** *For any  $\delta > 0$  we have  $c_n = O(|n|^{-1/2+\delta})$ .*

*Proof.* We use Satz 33 of Landau [5], p. 784 (with  $\alpha = 0$ ,  $\tau = 1$ ,  $\delta < 1/2$ ,  $\sigma = 1/2$ ) to deduce that

$$\eta_0(\gamma_n) = O(|n|^{1/2+\delta}), \quad \text{for all } \delta > 0.$$

The corollary is then clear from (6.8).

We conclude this section with a short table of the coefficients  $c_n$ .

TABLE 1

$n$	$\text{Re } c_n$	$\text{Im } c_n$	$ c_n $
0	1.5053	0	1.5053
1	-.0663	.0911	.1126
2	-.0927	-.1331	.1622
3	.0018	-.0031	.0035
4	.0352	.0116	.0370

The values were computed using the first 1,500,000 terms of (6.9) and the formula

$$\gamma_n \cdot \log 4 \cdot c_n = \sum_{k=1}^N \frac{a^*(k)}{k^{\gamma_n}} - \frac{s^*(N)}{(N+1)^{\gamma_n}} + \gamma_n \int_{N+1}^{\infty} \frac{s^*(x)}{x^{\gamma_n+1}} dx,$$

where  $N = 1.5 \times 10^6$ . The total error, due to roundoff and to the integral in this formula, is at most .002 in absolute value, and so  $c_n \neq 0$  for  $0 \leq n \leq 4$ .

**7. The cumulative distribution.** In this section we use the function  $\lambda(x)$  to show that the sequence  $\{s(n)/\sqrt{n}\}$  has no cumulative distribution function on the interval  $(\sqrt{3/5}, \sqrt{6})$ . Recall the following general definition.

**DEFINITION.** Let  $\{u_n\}$  be a sequence of real numbers contained in an interval  $J$ , and let  $\alpha \in J$ . If  $D(x, \alpha)$  denotes the number of  $n \leq x$  for which  $u_n \leq \alpha$ , and if the limit  $\lim_{x \rightarrow \infty} x^{-1} D(x, \alpha) = D(\alpha)$  exists, then the sequence  $\{u_n\}$  is said to have the distribution  $D(\alpha)$  at  $\alpha$ .  $D(\alpha)$  is called the cumulative distribution function of  $\{u_n\}$ .

**THEOREM 12.** *The cumulative distribution function of  $\{s(n)/\sqrt{n}\}$  does not exist at any point of  $(\sqrt{3/5}, \sqrt{6})$ .*

*Proof.* Let  $\alpha \in (\sqrt{3/5}, \sqrt{6})$ , and assume  $D(\alpha)$  exists for the sequence  $u_n = s(n)/\sqrt{n}$  in the above definition.

(a) We first show that  $D(\alpha)$  must equal one. By Corollary 1 to Theorem 5 we may choose an  $x_1 \in [1, 4]$  for which  $\lambda(x_1) < \alpha$ . Let  $\varepsilon$  be such that  $0 < \varepsilon < 1$  and

$$\lambda(x) < \alpha \quad \text{when } |x - x_1| \leq \varepsilon,$$

and set  $M = \max_{|x - x_1| \leq \varepsilon} \lambda(x)$ . Then  $M < \alpha$ . Set  $\delta = \alpha - M$ , and choose  $k_0$  so large that  $2^{-k_0}(x_1 - \varepsilon)^{-1/2} < \delta$ . From (2.15) we have for any  $x$  satisfying  $|x - x_1| \leq \varepsilon$  and for any  $k \geq k_0$  that  $|\lambda(x) - s(4^k x)/\sqrt{4^k x}| \leq 2^{-k} x^{-1/2} \leq 2^{-k_0}(x_1 - \varepsilon)^{-1/2} < \delta$ , so

$$\frac{s(4^k x)}{\sqrt{4^k x}} < \lambda(x) + \delta \leq M + (\alpha - M) = \alpha.$$

It follows that  $s(r)/\sqrt{r} < \alpha$  for every integer  $r$  of the interval  $4^k(x_1 - \varepsilon) < r \leq 4^k(x_1 + \varepsilon)$ . But the number of integers in this interval is

$$4^k(x_1 + \varepsilon) - 4^k(x_1 - \varepsilon) + O(1) = 2\varepsilon 4^k + O(1).$$



Thus if we put  $x_k^- = 4^k(x_1 - \varepsilon)$  and  $x_k^+ = 4^k(x_1 + \varepsilon)$ , we have

$$D(x_k^+, \alpha) = D(x_k^-, \alpha) + 2\varepsilon 4^k + O(1).$$

Dividing both sides by  $x_k^+ = (x_1 + \varepsilon)x_k^-/(x_1 - \varepsilon)$  and letting  $k \rightarrow \infty$  then gives that

$$D(\alpha) = D(\alpha) \frac{x_1 - \varepsilon}{x_1 + \varepsilon} + \frac{2\varepsilon}{x_1 + \varepsilon},$$

which implies  $D(\alpha) = 1$ , as claimed.

(b) We now show that  $D(\alpha) = 0$ ; this will contradict (a) and prove the theorem. We choose an  $x_1 \in [1, 4]$  with  $\lambda(x_1) > \alpha$ , and an  $\varepsilon$  for which

$$0 < \varepsilon < 1 \quad \text{and} \quad \lambda(x) > \alpha \quad \text{when} \quad |x - x_1| \leq \varepsilon.$$

We also pick  $k_0$  so that  $2^{-k_0}(x_1 - \varepsilon)^{-1/2} < \delta$ , where this time  $\delta = m - \alpha$  and  $m = \min_{|x - x_1| \leq \varepsilon} \lambda(x)$ . As before, we have for any  $x$  with  $|x - x_1| \leq \varepsilon$  and any  $k \geq k_0$  that

$$\left| \lambda(x) - \frac{s(4^k x)}{\sqrt{4^k x}} \right| < \delta,$$

whence

$$\frac{s(4^k x)}{\sqrt{4^k x}} > \lambda(x) - \delta \geq m - (m - \alpha) = \alpha.$$

Thus  $s(r)/\sqrt{r} > \alpha$  for all the integers  $r$  in the interval

$$4^k(x_1 - \varepsilon) < r \leq 4^k(x_1 + \varepsilon),$$

and

$$D(x_k^+, \alpha) = D(x_k^-, \alpha),$$

where  $x_k^+ = 4^k(x_1 + \varepsilon)$  and  $x_k^- = 4^k(x_1 - \varepsilon)$ . Therefore

$$\frac{1}{x_k^+} D(x_k^+, \alpha) = \frac{1}{x_k^-} D(x_k^-, \alpha) \cdot \frac{x_1 - \varepsilon}{x_1 + \varepsilon},$$

and letting  $k \rightarrow \infty$  shows that

$$D(\alpha) = D(\alpha) \cdot \frac{x_1 - \varepsilon}{x_1 + \varepsilon},$$

i.e. that  $D(\alpha) = 0$ . □

For the sequence  $u_n = t(n)/\sqrt{n}$  we have the analogous

**THEOREM 13.** *The cumulative distribution function of the sequence  $\{t(n)/\sqrt{n}\}$  does not exist at any point  $\alpha \in (0, \sqrt{3})$ . However it does exist when  $\alpha = 0$ , and  $D(0) = 0$ .*

*Proof.* The proof that  $D(\alpha)$  does not exist for  $\alpha$  in  $(0, \sqrt{3})$  follows, mutatis mutandis, the proof of Theorem 6. Thus assume that  $\alpha = 0$ . To show  $D(0) = 0$  we proceed as follows. Let  $n_\nu$  be the  $\nu$ th integer for which  $t(n) = 0$ . Clearly

$$\frac{1}{n}D(n, 0) \leq \frac{1}{n_\nu}D(n_\nu, 0) = \frac{\nu}{n_\nu}$$

if  $n_\nu \leq n < n_{\nu+1}$ , and so it suffices to show that

$$\nu/n_\nu \rightarrow 0 \quad \text{as } \nu \rightarrow \infty.$$

However, by the Vorbemerkung in Satz 13 of [1], if

$$\nu = \sum_{r=0}^k \epsilon_r 2^r$$

is the binary representation of  $\nu$ , then

$$n_\nu = \sum_{r=0}^k \epsilon_r 2^{2r+1} - 1.$$

Thus  $n_\nu \geq 2^{2k+1} - 1 > \frac{1}{2}\nu^2 - 1$ , and so  $\nu/n_\nu \leq 2\nu/(\nu^2 - 2) \rightarrow 0$  as  $\nu \rightarrow \infty$ .  $\square$

As the above proofs show, the nonexistence of the cumulative distribution functions is attributable to the fact that the sequences  $s(n)/\sqrt{n}$  and  $t(n)/\sqrt{n}$  behave very “sluggishly”.

**8. The logarithmic distribution.** It is possible to show that a modified distribution function does exist for the sequences  $\{s(n)/\sqrt{n}\}$  and  $\{t(n)/\sqrt{n}\}$ . The type of distribution we consider is defined as follows.

**DEFINITION.** Let  $\{u_n\}$  be a real sequence contained in an interval  $J$ , and let  $\alpha \in J$ . If

$$L(x, \alpha) = \sum_{\substack{1 \leq n \leq x \\ u_n \leq \alpha}} \frac{1}{n},$$

and if the limit

$$\lim_{x \rightarrow \infty} \frac{1}{\log x} L(x, \alpha) = L(\alpha)$$

exists, then the sequence  $\{u_n\}$  is said to have the logarithmic distribution  $L(\alpha)$  at  $\alpha$ .  $L(\alpha)$  is called the logarithmic distribution function of the sequence.

We shall prove that both sequences  $\{s(n)/\sqrt{n}\}$  and  $\{t(n)/\sqrt{n}\}$  have logarithmic distribution functions which are defined everywhere in the respective intervals  $[\sqrt{3/5}, \sqrt{6}]$  and  $[0, \sqrt{3}]$ . We need a lemma.

**LEMMA 7.** *Let  $\alpha \in [\sqrt{3/5}, \sqrt{6}]$  be fixed and let  $S_\alpha$  denote the set  $S_\alpha = \{x: 1 \leq x \leq 4 \text{ and } \lambda(x) = \alpha\}$ . Then  $S_\alpha$  has measure zero.*

*Proof.* Let  $x_0 = \sum_{r=0}^{\infty} d_r 4^{-r}$  be an element of  $S_\alpha$  which is normal to the base 4. Choose an  $n \geq 1$  for which  $d_j = 0$  for  $n \leq j \leq n+3$ , and set

$$x_n = x_0 + 4^{-n}, \quad y_n = x_0 + 4^{-n} + 4^{-n-3} = x_0 + h_n.$$

As in the proof of Theorem 6 we have that  $|a(x_0) - a(x_n)| = 2^{-n}$ . Now if  $x^*$  satisfies  $x_n < x^* < y_n$ , it is easy to see that  $x^* = \sum_{r=0}^{\infty} d_r^* 4^{-r}$ , with

$$d_r^* = \begin{cases} d_r, & r < n, \\ 1, & r = n, \\ 0, & r = n+1, n+2. \end{cases}$$

Thus

$$\begin{aligned} |a(x^*) - a(x_n)| &= \left| \sum_{r=n+3}^{\infty} \frac{\rho(d_r) a(b_r) - \rho(d_r^*) a(b_r^*)}{2^r} \right| \\ &\leq \sum_{r=n+3}^{\infty} 2^{1-r} = 2^{-n-1}, \end{aligned}$$

and it follows that

$$\begin{aligned} |a(x_0) - a(x^*)| &= |a(x_0) - a(x_n) + a(x_n) - a(x^*)| \\ &\geq 2^{-n} - 2^{-n-1} = 2^{-n-1}. \end{aligned}$$

Furthermore, equation (5.7) implies

$$\begin{aligned} |\lambda(x_0) - \lambda(x^*)| &= \left| \frac{a(x^*) - a(x_0)}{\sqrt{x^*}} + O(|x^* - x_0|) \right| \\ &\geq \kappa_0 2^{-n} - \kappa_1 4^{-n} \geq \kappa_2 2^{-n}, \quad \text{for } n \geq n_0, \end{aligned}$$

where  $\kappa_0, \kappa_1, \kappa_2$  are positive constants and  $n_0$  is sufficiently large. Therefore, for  $n \geq n_0$  satisfying  $d_n = d_{n+1} = d_{n+2} = d_{n+3} = 0$ , we have

$$\lambda(x^*) \neq \alpha \quad \text{for } x_n < x^* < x_n + 4^{-n-3}.$$

If  $m$  denotes Lebesgue measure, we deduce

$$(8.1) \quad \frac{1}{h_n} m(S_\alpha \cap (x_0, x_0 + h_n)) \leq \frac{4^{-n}}{h_n} = \frac{64}{65} < 1,$$

for an infinite sequence of  $h_n$ 's tending to zero.

On the other hand, if  $\chi_\alpha$  denotes the characteristic function of the set  $S_\alpha$ , then

$$(8.2) \quad \begin{aligned} \frac{1}{h} m(S_\alpha \cap (x_0, x_0 + h)) \\ = \frac{1}{h} \int_{x_0}^{x_0+h} \chi_\alpha(t) dt \rightarrow \chi_\alpha(x_0), \quad \text{as } h \rightarrow 0, \end{aligned}$$

for almost all  $x_0$  (see [4], p. 173). Equation (8.1) shows therefore that all normal numbers  $x_0$  in  $S_\alpha$  lie in the null set of exceptional numbers for which (8.2) does not hold, since for these  $x_0$ ,  $\chi_\alpha(x_0) = 1$ . But this implies  $m(S_\alpha) = 0$ .  $\square$

The argument in the above lemma is due to A. J. E. M. Janssen (private communication).

We can now prove

**THEOREM 14.** *If  $\alpha \in [\sqrt{3/5}, \sqrt{6}]$ , then the logarithmic distribution function of the sequence  $\{s(n)/\sqrt{n}\}$  exists at  $\alpha$ , and has the value*

$$(8.3) \quad L(\alpha) = \frac{1}{\log 4} \int_{E_\alpha} \frac{1}{x} dx,$$

where  $E_\alpha$  is the set

$$(8.4) \quad E_\alpha = \{x: 1 \leq x \leq 4 \text{ and } \lambda(x) \leq \alpha\}.$$

*Proof.* Let  $I_k$  denote the set of integers  $r$  contained in the interval  $4^k \leq r < 4^{k+1}$ ,  $k \geq 0$ , and consider the sum

$$\sigma_k(\alpha) = \sum_{\substack{r \in I_k \\ \lambda(r) \leq \alpha}} \frac{1}{r} = \sum_{r=4^k}^{4^{k+1}-1} \frac{\omega_\alpha(r/4^k)}{r},$$

where  $\omega_\alpha$  is the characteristic function of the set  $E_\alpha$ . Note that  $\sigma_k(\alpha)$  is just a Riemann sum for the function  $\omega_\alpha(x)/x$  on the interval  $[1, 4]$ , since

$$\sigma_k(\alpha) = \sum_{r=4^k}^{4^{k+1}-1} \frac{\omega_\alpha(r/4^k)}{r/4^k} 4^{-k}.$$

Now  $\lambda$  is a continuous function, so it is clear from (8.4) that the discontinuities of  $\omega_\alpha$  are contained in the set  $S_\alpha = \{x: 1 \leq x \leq 4 \text{ and } \lambda(x) = \alpha\}$ . By Lemma 7,  $S_\alpha$  has measure zero, and therefore  $\omega_\alpha$  is Riemann integrable. (See [4], p. 64.) Consequently,

$$(8.5) \quad h(\alpha) = \lim_{k \rightarrow \infty} \sigma_k(\alpha) = \int_1^4 \frac{\omega_\alpha(x)}{x} dx = \int_{E_\alpha} \frac{1}{x} dx.$$

Note also that  $h(\alpha)$  is a continuous function of  $\alpha$ , since the set  $E_{\alpha+\varepsilon}$  tends to the set  $E_\alpha$  as  $\varepsilon \rightarrow 0^+$ , and since  $E_{\alpha-\varepsilon}$  tends to  $E_\alpha - S_\alpha$  as  $\varepsilon \rightarrow 0^+$ , which differs from  $E_\alpha$  by the null set  $S_\alpha$ .

This fact implies easily that

$$(8.6) \quad \lim_{k \rightarrow \infty} \sigma_k(\alpha - 2^{-k}) = \lim_{k \rightarrow \infty} \sigma_k(\alpha + 2^{-k}) = h(\alpha).$$

For instance, if  $k_0$  is fixed and  $k \geq k_0$ , we have

$$\sigma_k(\alpha - 2^{-k_0}) \leq \sigma_k(\alpha - 2^{-k}) \leq \sigma_k(\alpha + 2^{-k}) \leq \sigma_k(\alpha + 2^{-k_0}).$$

Thus by (8.5),

$$\begin{aligned} h(\alpha - 2^{-k_0}) &\leq \liminf_{k \rightarrow \infty} \sigma_k(\alpha - 2^{-k}) \leq \limsup_{k \rightarrow \infty} \sigma_k(\alpha + 2^{-k}) \\ &\leq h(\alpha + 2^{-k_0}). \end{aligned}$$

But for large  $k_0$ , both sides of this inequality can be made arbitrarily close to  $h(\alpha)$ , and this proves (8.6).

We now show that the limit of

$$\bar{\sigma}_k(\alpha) = \sum_{\substack{r \in I_k \\ s(r) \leq \alpha\sqrt{r}}} \frac{1}{r},$$

as  $k \rightarrow \infty$ , is  $h(\alpha)$ . From (2.15) we have

$$\left| \lambda(r) - \frac{s(r)}{\sqrt{r}} \right| \leq \frac{1}{\sqrt{r}} \leq 2^{-k}, \quad \text{for } r \in I_k,$$

and so

$$\begin{aligned} \frac{s(r)}{\sqrt{r}} \leq \alpha & \quad \text{implies } \lambda(r) \leq \alpha + 2^{-k}, \\ \lambda(r) \leq \alpha - 2^{-k} & \quad \text{implies } \frac{s(r)}{\sqrt{r}} \leq \alpha, \end{aligned}$$

for these  $r$ . It follows that

$$\sigma_k(\alpha - 2^{-k}) \leq \bar{\sigma}_k(\alpha) \leq \sigma_k(\alpha + 2^{-k}).$$

Letting  $k \rightarrow \infty$  and using (8.6) then gives

$$\lim_{k \rightarrow \infty} \bar{\sigma}_k(\alpha) = h(\alpha).$$

Thus we have also

$$(8.7) \quad \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{k=0}^{m-1} \bar{\sigma}_k(\alpha) = h(\alpha),$$

since the  $(C, 1)$  method is regular.

Finally, suppose that  $n \geq 1$  is arbitrary and  $m$  is chosen so that  $4^m \leq n < 4^{m+1}$ . Then  $m = [\log n / \log 4]$ , and

$$\frac{m}{\log n} \cdot \frac{1}{m} \sum_{k=0}^{m-1} \bar{\sigma}_k(\alpha) \leq \frac{1}{\log n} \sum_{\substack{r=1 \\ s(r) \leq \alpha \sqrt{r}}}^n \frac{1}{r} \leq \frac{m+1}{\log n} \cdot \frac{1}{m+1} \sum_{k=0}^m \bar{\sigma}_k(\alpha).$$

Hence by (8.7),

$$\lim_{n \rightarrow \infty} \frac{1}{\log n} \sum_{\substack{r=1 \\ s(r) \leq \alpha \sqrt{r}}}^n \frac{1}{r} = \frac{1}{\log 4} h(\alpha) = \frac{1}{\log 4} \int_{E_\alpha} \frac{1}{x} dx,$$

and this proves (8.3).  $\square$

**THEOREM 15.** *If  $\alpha \in [0, \sqrt{3}]$ , then the logarithmic distribution function of the sequence  $\{t(n)/\sqrt{n}\}$  exists at  $\alpha$ , and has the value*

$$L^*(\alpha) = \frac{1}{\log 4} \int_{E_\alpha^*} \frac{1}{x} dx,$$

where  $E_\alpha^* = \{x: 1 \leq x \leq 4 \text{ and } \mu(x) \leq \alpha\}$ .

*Proof.* The theorem is proved by exactly the same argument used to prove Theorem 14, the crucial point being that the set  $S_\alpha^* = \{x: 1 \leq x \leq 4 \text{ and } \mu(x) = \alpha\}$  has measure zero. We omit the details.

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